Ricci Flow for Warped Product Manifolds

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May 04, 2010

Outline

Ricci Flow: definition

- Ricci flow is a geometric flow defined on a manifold $\mathcal{M}$ with a metric g_{ij}. It deforms the metric along a parameter λ according to the differential equation–

$$\frac{\partial g_{ij}}{\partial \lambda} = -2R_{ij} \tag{1}$$

- To preserve the volume the Ricci flow can be normalized to give –

$$\frac{\partial g_{ij}}{\partial \lambda} = -2R_{ij} + \frac{2}{n}\langle R \rangle g_{ij} \tag{2}$$

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- RF is like the heat equation and tends to smooth out the irregularities in the metric.
- Finding the best metric on a manifold to solve mathematical problems like the Poincare Conjecture. (Perelman [4])
- Renormalization Group flow in non-linear σ-model of string theory leads to Ricci flow in the lowest order.

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RF as heat flow

Conformally flat 2-d manifold – $ds^2 = e^{2\phi(x,y)}\left(dx^2 + dy^2\right)$ with Ricci curvature –

$$R_{xx} = R_{yy} = -\left(\partial_x^2\phi + \partial_y^2\phi\right) \tag{3}$$

and the Ricci flow Eq.(1) becomes

$$\frac{\partial\phi}{\partial\lambda} = \triangle\phi \tag{4}$$

where $\triangle = e^{-2\phi}\left(\partial_x^2 + \partial_y^2\right)$ is generalized Laplacian.

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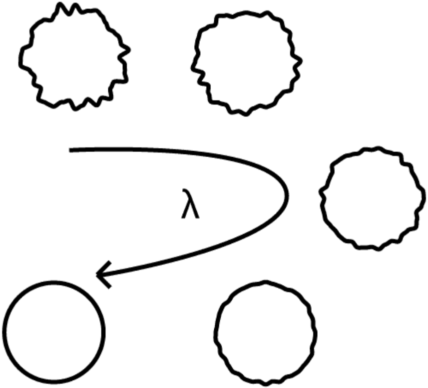
Sphere

- metric: $ds^2 = r^2(\lambda)\left(d\theta^2 + \sin^2\theta d\phi^2\right)$
- RF: $r \sim (\lambda_0 - \lambda)^{1/2}$ NRF: $r =$ constant
- but if $r = r_0(\lambda) + r_1$ with $r_1 = Y^l_m(\theta, \phi)$ then $r_1 \sim e^{-l(l+1)\lambda}$



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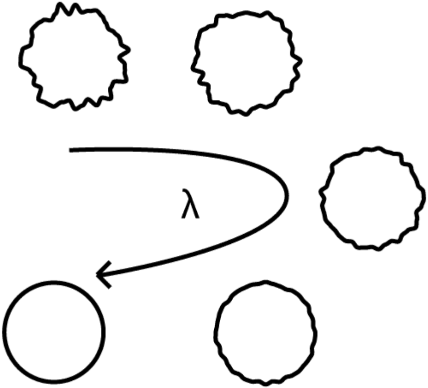

Warped manifolds

- Extra-dimensional brane world metric

$$ds^2 = e^{2f(\sigma,\lambda)}\eta_{\mu\nu}dx^\mu dx^\nu + r_c^2(\sigma,\lambda)d\sigma^2 \tag{5}$$

- The RF is now a system of PDEs –

$$\dot{f} = \frac{1}{r_c^2}\left(f'' + 4f'^2 - \frac{f' r_c'}{r_c}\right) \tag{6}$$

$$\dot{r}_c = \frac{4}{r_c}\left(f'' + f'^2 - \frac{f' r_c'}{r_c}\right) \tag{7}$$

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Separable solution

- Assume separable functions – $r_c(\sigma, \lambda) = r_c(\lambda)$ $f(\sigma, \lambda) = f_\sigma(\sigma) + f_\lambda(\lambda)$
- The equations become separable and the solution becomes – $ds^2 = \left(1 + \frac{\lambda}{\lambda_c}\right)\left[\exp\left(\pm\frac{\sigma}{\sqrt{2\lambda_c}}\right)\eta_{\mu\nu}dx^\mu dx^\nu + d\sigma^2\right]$
- Curvature scalar $R = -\frac{5/2}{\lambda+\lambda_c}$. Flow becomes singular at $\lambda = -\lambda_c$.

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Scaling solution

- Invariance under $\sigma \to \alpha\sigma$ and $\lambda \to \alpha^2\lambda$. So use variable $x = \frac{1}{2} \ln \frac{\lambda}{\sigma^2}$ and convert to ODE.
- Also use $B = \frac{e^x}{r_c}$

$$\frac{df}{dx} = A \tag{8}$$

$$\frac{dA}{dx} = \frac{A}{2B^2} - A - 24A^3B^2 \tag{9}$$

$$\frac{dB}{dx} = -4AB + B + 24A^2B^3 \tag{10}$$

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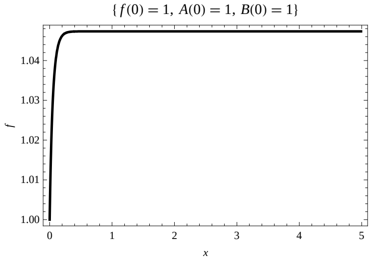
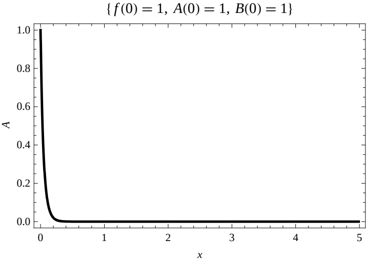
Scaling solution: plot

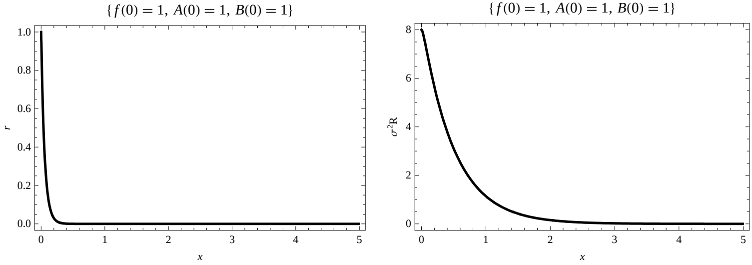


(a) f v/s x



(b) A v/s x

Scaling solution: plot



(c) r_c v/s x

(d) $\sigma^2 R$ v/s x

Scaling solution: singularities

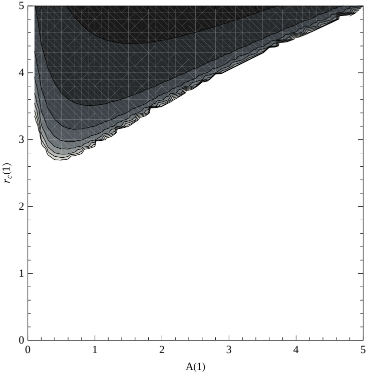


Figure: phase diagram showing non-singular and singular flows in space of ($A(0)$, $r_c(0)$)

RG flow

- Behaviour of non-linear σ-models under renormalization given by –

$$\frac{\partial g_{ij}}{\partial \lambda} = -\beta_{ij} \tag{11}$$

- Perturbative expansion of β in terms of α' *the inverse string tension*

$$\beta_{ij} = \alpha' \beta_{ij}^{(1)} + \alpha'^2 \beta_{ij}^{(2)} + \alpha'^3 \beta_{ij}^{(3)} + \alpha'^4 \beta_{ij}^{(4)} + O\left(\alpha'^5\right) \tag{12}$$

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β-functions

β functions upto $O(\alpha'^4)$ (for explicit expressions see[6, 7])

$$\beta_{ij}^{(1)} = R_{ij} \qquad \beta_{ij}^{(2)} = \frac{1}{2} R_{iklm} R_j{}^{klm} \tag{13a}$$

$$\begin{aligned} \beta_{ij}^{(3)} = & \frac{1}{8} \nabla_p R_{iklm} \nabla^p R_j{}^{klm} - \frac{1}{16} \nabla_i R_{klmp} \nabla_j R^{klmp} \\ & + \frac{1}{2} R_{klmp} R_i{}^{mlr} R_j{}^{kp}{}_r - \frac{3}{8} R_{iklj} R^{kspr} R^l{}_{spr} \end{aligned} \tag{13b}$$

$$\begin{aligned} \beta_{ij}^{(4)} = & -\frac{1}{16} R_1 + \frac{1}{48} R_2 - \frac{1}{16} \left(\frac{1}{2} + \zeta(3) \right) R_3 + \frac{1}{4} \left(1 + \zeta(3) \right) R_4 \\ & + \frac{1}{16} \left(\frac{13}{3} - 3\zeta(3) \right) R_5 + \frac{1}{8} \left(\frac{2}{3} - \zeta(3) \right) R_6 + \frac{1}{4} \left(\frac{8}{3} + \zeta(3) \right) R_7 \\ & + \frac{1}{4} \left(-\frac{1}{3} + \zeta(3) \right) R_8 + \frac{1}{12} R_9 + \frac{1}{12} R_{10} - \frac{1}{6} R_{11} \\ & + \frac{1}{16} \left(\frac{4}{3} + \zeta(3) \right) R_{12} - \frac{1}{4} \left(\frac{4}{3} + \zeta(3) \right) R_{13} + \text{higher derivatives} \end{aligned}$$

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- Curvature scalar $R = -\frac{4}{r^2}\left[2f'' + 5f'^2\right] = -\frac{20k^2}{\Omega}$.
- Constant negative curvature i.e. Anti deSitter (AdS) space time. Brane world model of Randall-Sundrum. [5]

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ODE for Ω

- Single ODE left for scale factor Ω

$$\frac{1}{8k^2}\frac{d\Omega}{d\lambda} = 1 - \frac{\alpha' k^2}{\Omega} + 2\left(\frac{\alpha' k^2}{\Omega}\right)^2 - \frac{3+5\zeta(3)}{2}\left(\frac{\alpha' k^2}{\Omega}\right)^3$$

- rescale the variables as $\bar{\Omega} = \frac{\Omega}{|\alpha'|k^2}$; $\bar{\lambda} = \frac{8\lambda}{|\alpha'|}$
- rescaled equation will be

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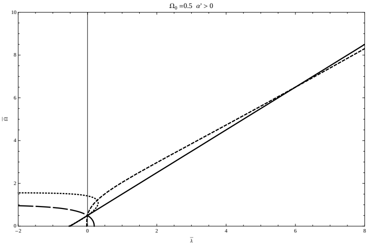
Solutions

- $\bar{\lambda} + C_1 = \bar{\Omega}$
- $\bar{\lambda} + C_2 = \bar{\Omega} \pm \ln \left|\bar{\Omega} \mp 1\right|$ or $\bar{\Omega} = 1$
- $\bar{\lambda} + C_3 = \bar{\Omega} \pm \frac{1}{2} \ln \left|\bar{\Omega}^2 \mp \bar{\Omega} + 2\right| - \frac{3}{\sqrt{7}} \tan^{-1} \left(\frac{2\bar{\Omega} \mp 1}{\sqrt{7}}\right)$
- $\bar{\lambda} + C_4 =$
  $\bar{\Omega} \pm a \ln \left|\bar{\Omega} \mp \xi_4\right| \mp \frac{b}{2} \ln \left|\bar{\Omega}^2 \pm \beta\bar{\Omega} + \gamma\right| - \frac{2c - \beta b}{\sqrt{4\gamma - \beta^2}} \tan^{-1} \left(\frac{2\bar{\Omega} \pm \beta}{\sqrt{4\gamma - \beta^2}}\right)$ or
  $\bar{\Omega} = \xi_4$ where

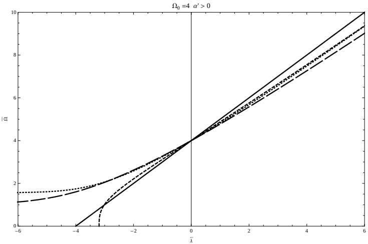
$$\xi_4 \approx 1.5636 \quad \beta \approx 0.5636 \quad \gamma \approx 2.8812 \tag{14a}$$

$$a \approx 0.6158 \quad b \approx -0.3841 \quad c \approx 1.7464 \tag{14b}$$

Solutions: plot $\alpha' > 0$



(a) $\bar{\Omega}_0 = 0.5$



(b) $\bar{\Omega}_0 = 4$

Solutions: expansion

- Solutions for small curvature (large $\bar{\Omega}$)

$$\lambda + C_1 = \bar{\Omega} \tag{15a}$$

$$\lambda + C_2 = \bar{\Omega} \pm \ln \bar{\Omega} - \left(\frac{1}{\bar{\Omega}}\right) \mp \frac{1}{2}\left(\frac{1}{\bar{\Omega}}\right)^2 - \frac{1}{3}\left(\frac{1}{\bar{\Omega}}\right)^3 + \dots \tag{15b}$$

$$\lambda + \tilde{C}_3 = \bar{\Omega} \pm \ln \bar{\Omega} + \left(\frac{1}{\bar{\Omega}}\right) \pm \frac{3}{2}\left(\frac{1}{\bar{\Omega}}\right)^2 + \frac{1}{3}\left(\frac{1}{\bar{\Omega}}\right)^3 + \dots \tag{15c}$$

$$\lambda + \tilde{C}_4 = \bar{\Omega} \pm \ln \bar{\Omega} + \left(\frac{1}{\bar{\Omega}}\right) \mp 0.7526\left(\frac{1}{\bar{\Omega}}\right)^2 - 2.6699\left(\frac{1}{\bar{\Omega}}\right)^3 + \dots \tag{15d}$$

- leading order correction $\sim \ln \bar{\Omega}$

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Conclusions

- Conformally AdS spacetime is a solution to RG flow equations upto 4th order in α'. This is same as the spacetime in the brane world model of the Universe proposed by Randall and Sundrum. [5]
- Apart from flat space, two *soliton* solutions exist at orders 2 and 4. But these have large curvature scales and are non-perturbative effects.
- Higher order terms in the RG flow provide a leading order correction of $\sim \ln \bar{\Omega}$, while other correction vanish in the limit.

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Possible Further Work

- Extension of scaling type solutions to higher order terms in Renormalization Group flow.
- Proving that AdS spacetime is a solution in all orders in α'. But higher order terms have not been computed beyond order 4.
- Nature of solutions, solitons for even higher order expansion...
- Studying such properties from general non-perturbative principles.

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Q & A

Thank You